Def (α - Hölder)

Let f be an integrable function on the circle and $0 < \alpha \leq 1$. f is α - Hölder if $\exists\ C > 0$ s.t.

$$|f(x+h) - f(x)| < C|h|^{\alpha}, \quad \forall\ x, h.$$

(Riemann - Lebesgue Lemma)

If f is integrable, then $\hat{f}(n) \to 0$ as $n \to \infty$

Q : If f is α - Hölder, can we say something about the order of decay of $\hat{f}(n)$?

- If f is α - Hölder, then $\hat{f}(n) = O\left(\frac{1}{|n|^{\alpha}}\right)$.
- Give a counterexample to show this decay rate cannot be improved.

- If f is α-Hölder with $\alpha > \frac{1}{2}$, then its Fourier series is absolutely convergent.

- If f is α-Hölder, then $\hat{f}(n) = O\left(\frac{1}{|n|^\alpha}\right)$.

Pf: $\vdash$: $\hat{f}(n) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+\frac{\pi}{n})e^{-inx}\,dx$

check: $\hat{f}(n) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx$

$x = y + \frac{\pi}{n}$

$$= \frac{1}{2\pi}\int_{-\pi-\frac{\pi}{n}}^{\pi-\frac{\pi}{n}} f(y+\tfrac{\pi}{n})e^{-in(y+\frac{\pi}{n})}\,dy$$

$e^{-in\cdot\frac{\pi}{n}} = e^{-i\pi} = -1$

$$= \frac{-1}{2\pi}\int_{-\pi-\frac{\pi}{n}}^{\pi-\frac{\pi}{n}} f(y+\tfrac{\pi}{n})e^{-iny}\,dy$$

$f(y+\frac{\pi}{n})e^{-iny}$ is 2π-periodic

$$= -\frac{1}{2\pi}\int_{-\pi}^{\pi} f(y+\tfrac{\pi}{n})e^{-iny}\,dy.$$

$$|\hat{f}(n)| = \frac{1}{2}|\hat{f}(n) + \hat{f}(n)|$$

$$= \frac{1}{2}\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+\tfrac{\pi}{n})e^{-inx}\,dx\right|$$

$$= \frac{1}{4\pi}\int_{-\pi}^{\pi} |f(x) - f(x+\tfrac{\pi}{n})|\,dx$$

$$\leq \frac{1}{4\pi}\int_{-\pi}^{\pi} C\left(\frac{\pi}{n}\right)^{\alpha} dx$$

$$= \frac{C\pi^{\alpha}}{2}\left(\frac{1}{n^{\alpha}}\right). \qquad \square$$

- In general, we cannot get $\hat{f}(n) = o\left(\frac{1}{|n|^{\alpha}}\right)$ even if f is α-Hölder.

Let $f(x) := \sum_{k=0}^{\infty} 2^{-k\alpha} e^{i2^k x}$, $\forall\ 0<\alpha<1$.

We will show (1) f is α-Hölder

(2) $\hat{f}(2^n) = \frac{1}{(2^n)^{\alpha}}$

Pf: $$|f(x+h) - f(x)| = \left|\sum_{k=0}^{\infty} 2^{-k\alpha} e^{i2^k x}(e^{i2^k h} - 1)\right|$$

$$\leq \sum_{k=0}^{\infty} 2^{-k\alpha} |e^{i2^k h} - 1|$$

$$= \sum_{k=0}^{\infty} 2^{-k\alpha} |e^{-i2^{k-1} h}||e^{i2^k h} - 1|$$

$$= \sum_{k=0}^{\infty} 2^{-k\alpha} |e^{i2^{k-1} h} - e^{-i2^{k-1} h}|$$

$$= \sum_{k=0}^{\infty} 2^{-k\alpha} |2\sin 2^{k-1} h|.$$

The idea is

when k is small, $|2^{k-1}h|$ is small, we use $|\sin x| \le |x|$.

When k is large, $2^{-k\alpha}$ is small, we use $|\sin x| \le 1$.

Take $N \in \mathbb{N}$ s.t. $2^{N-1} \le \frac{1}{|h|} < 2^N$.

$$\sum_{k=0}^{\infty} 2^{-k\alpha} |\sin 2^{k-1} h|$$

$$= \sum_{k=0}^{N-1} 2^{-k\alpha} |\sin 2^{k-1} h| + \sum_{k=N}^{\infty} 2^{-k\alpha} |\sin 2^{k-1} h|$$

$$\le \sum_{k=0}^{N-1} 2^{-k\alpha} 2^{k-1} |h| + \sum_{k=N}^{\infty} 2^{-k\alpha}$$

$$= |h| \sum_{k=0}^{N-1} 2^{(1-\alpha)k} + \sum_{k=N}^{\infty} 2^{-k\alpha}$$

$$= |h| \frac{2^{N(1-\alpha)} \cancel{-1}}{2^{1-\alpha} - 1} + \frac{2^{-N\alpha}}{1 - 2^{1-\alpha}}$$

$2^N \approx |h|^{-1}$

$2^N \le 2|h|^{-1}$

$$\le |h| \frac{(2|h|^{-1})^{1-\alpha}}{2^{1-\alpha} - 1} + \frac{|h|^{\alpha}}{1 - 2^{1-\alpha}}$$

$2^{-N} \le |h|$

$$= C_\alpha |h|^\alpha$$

$$\hat{f}(2^n) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{k=0}^{\infty} 2^{-k\alpha} e^{i2^k x}\right) e^{-i2^n x}\,dx$$

$$= \frac{1}{2\pi}\sum_{k=0}^{\infty} 2^{-k\alpha}\int_{-\pi}^{\pi} e^{i(2^k-2^n)x}\,dx$$

$$= \frac{1}{2\pi}\cdot 2^{-n\alpha}\cdot 2\pi = \frac{1}{(2^n)^{\alpha}}$$

$\square$

- If f is α-Hölder with $\alpha > \frac{1}{2}$, then

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty.$$

Step 1: $\displaystyle\sum_{n=-\infty}^{\infty} |\sin nh|^2 |\hat{f}(n)|^2 \le C^2 h^{2\alpha}$

Step 2: $\displaystyle\sum_{2^{p-1}<|n|\le 2^p} |\hat{f}(n)|^2 \le 2C^2\left(\frac{\pi}{2^{p+1}}\right)^{2\alpha}$, $\forall\, p \in \mathbb{N}$.

Step 3: $\displaystyle\sum_{2^{p-1}<|n|\le 2^p} |\hat{f}(n)| \le C_\alpha \frac{1}{2^{p(\alpha-\frac{1}{2})}}$.

Pf of Step 1: $f_h(x) = f(x+h)$.

$$\hat{f_h}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+h)e^{-inx}\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi+h}^{\pi+h} f(x)e^{-inx}e^{inh}\,dx$$

$$= e^{inh} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\, dx$$

$$= e^{inh} \hat{f}(n).$$

Parseval Identity

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |\underbrace{f(x+h)}_{f_h(x)} - \underbrace{f(x-h)}_{f_{-h}(x)}|^2 dx \overset{\text{Parseval Identity}}{=} \sum_{n=-\infty}^{\infty} |\hat{f_h}(n) - \hat{f}_{-h}(n)|^2$$

$$= \sum_{n=-\infty}^{\infty} |e^{inh} - e^{-inh}|^2 |\hat{f}(n)|^2$$

$$= 4 \sum_{n=-\infty}^{\infty} |\sin nh|^2 |\hat{f}(n)|^2$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x+h) - f(x-h)|^2 dx \le \frac{1}{2\pi}\int_{-\pi}^{\pi} (C|2h|^{\alpha})^2 dx$$

$$= 4C^2|h|^{2\alpha}.$$

Step 2: Choose $h = \frac{\pi}{2^{p+1}}$.

When $2^{p-1} < |n| \le 2^p$, $\frac{\pi}{4} < |nh| \le \frac{\pi}{2}$.

Then $|\sin nh| > \frac{1}{\sqrt{2}} \Rightarrow |\sin nh|^2 > \frac{1}{2}$.

$$\frac{1}{2}\sum_{2^{p-1}<|n|\le 2^p} |\hat{f}(n)|^2 \le \sum_{2^{p-1}<|n|\le 2^p} |\sin nh|^2 |\hat{f}(n)|^2 \le C^2|h|^{2\alpha}.$$

$$= C^2\left(\frac{\pi}{2^{p+1}}\right)^{2\alpha}$$

Step 3: $\displaystyle\sum_{2^{p-1}<|n|\le 2^p} |\hat{f}(n)| = \sum_{2^{p-1}<|n|\le 2^p} |\hat{f}(n)|\cdot 1$

Cauchy-Schwarz

$$\le \sqrt{\sum_{2^{p-1}<|n|\le 2^p} 1^2}\sqrt{\sum_{2^{p-1}<|n|\le 2^p} |\hat{f}(n)|^2}$$

$$\le 2^{\frac{p-1}{2}} C\cdot\left(\frac{\pi}{2^{p+1}}\right)^{\alpha}$$

$$= C_\alpha 2^{p(\frac{1}{2}-\alpha)}$$

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| = \sum_{p=1}^{\infty}\sum_{2^{p-1}<|n|\le 2^p} |\hat{f}(n)|$$

$$\le \sum_{p=1}^{\infty} C_\alpha 2^{p(\frac{1}{2}-\alpha)}$$

$$= C_\alpha \sum_{p=1}^{\infty}\left(2^{\frac{1}{2}-\alpha}\right)^p$$

Since $\alpha > \frac{1}{2}$, $2^{\frac{1}{2}-\alpha} < 1$.

Therefore, $\displaystyle\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty$.

Hence, the Fourier series of f is absolutely convergent. $\square$